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NUMERICAL EXPERIENCES CONCERNING THE CALCULATION OF PERTURBATION IN CELESTIAL MECHANICS WITH THE METHOD OF SPINOR REGULARIZATION

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Numerical Experiences Concerning the Calculation of Perturbation in Celestial Mechanics with the Method of Spinor Regularization

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1. Theoretical Principles.

The plane motion of vehicle under the influence of the gravitation of a single central body can, according to LEVO-CIVITA /1/ [1] be regularized by replacing the Cartesian coordinates x_1, x_2 of the plane motion at the zero point of which the central body is situated by parameters u_1, u_2, and the time by a transformed time according to transformation equations

$$x_1 = u_1^2 - u_2^2 , \quad x_2 = 2 u_1 u_2 , \quad dt = r\, ds , \tag{1}$$

where $r = (x_1^2 + x_2^2)^{1/2} = u_1^2 + u_2^2$. Here is proportional to the increase of eccentric anomaly of the vehicle.

If the vehicle moves in 3 dimensions, the regularization of

1) Numbers in brackets refer to the bibliography, page 402 (not translated)

LEVI-CIVITA can be generalized /2/. In the following text we compile the formulas of this method in the manner in which they have been specially developed for the calculation of perturbation in /3/.

The physical space of coordinates x_1, x_2, x_3, is mapped on a 4-dimensional u_1, u_2, u_3, u_4 according to the provision

$$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2, \quad x_2 = 2(u_1 u_2 - u_3 u_4), \quad x_3 = 2(u_1 u_3 + u_2 u_4). \quad (2)$$

For the radius vector we obtain

$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2} = u_1^2 + u_2^2 + u_3^2 + u_4^2.$$

Due to the difference of the dimensions of the x_i, and the u_i space, the values of u_i must satisfy the secondary condition

$$u_4\,du_1 - u_3\,du_2 + u_2\,du_3 - u_1\,du_4 = 0. \quad (3)$$

This is the "spinor regularization" (in the following text called the KS method) mentioned in the title.

The central body with a mass of M is located in the origin of the x_i space and the vehicle with vanishing mass is located at point $r = (x_1, x_2, X_3)$. In addition to the attraction of the central body, a perturbing acceleration S shall effect the central body. Then the equations of motion for the vehicle are

$$\ddot{\mathbf{r}} = -\frac{M}{r^3}\,\mathbf{r} + \mathbf{S}\,, \tag{4}$$

if one point describes the derivative with respect to t. The singularity of the differential equation (4) in the origin disappears in transformation by virtue of the transition toward the above-described u_i space and an appropriate new time variable s. Thus

$$u_i'' + \Omega^2 u_i = \frac{r}{4}\,S_i + \frac{A}{2}\,u_i \quad (i = 1, 2, 3, 4) \tag{5}$$

or

$$u_i'' + \Omega^2 u_i = \frac{1}{2M}\left(\Omega^2 r + \sum_{j=1}^{4} (u_j')^2\right)\left(r\,S_i + \frac{u_i}{\Omega^2}\sum_{j=1}^{4} S_j\,u_j'\right) \quad (i = 1, 2, 3, 4)\,. \tag{6}$$

is obtained (for details see /3/.)

In (5)

$$dt = r\,ds = (u_1^2 + u_2^2 + u_3^2 + u_4^2)\,ds\,,$$

a prime designates differentiation with respect to s, furthermore

$$\Omega^2 = \frac{M}{4a}\,,$$

where a denotes the large semiaxis of the unperturbed vehicle orbit which is osculatory at the time t = 0, S_i are the components of the perturbing acceleration in the u_i space, and

$$A = \int \sum_{j=1}^{4} S_j \, du_j' \equiv \text{work of the perturbing acceleration.}$$

In (6) the time is transformed somewhat differently:

$$dt = r \left[\frac{2}{M} \left(r\, \Omega^2 + \sum_{j=1}^{4} (u_j')^2 \right) \right]^{1/2} ds. \tag{7}$$

The remaining designations are the same as in (5)

Expression (5) is simpler, but requires an additional integration (for A); (6) fails in the case of a parabolic orbit of the unperturbed mobil vehicle, since in that case the denominator disappears.

The method of the perturbation theory consists in analytically computing the unperturbed orbit of the vehicle, osculatory at the time t = 0 (designated for brevity in the following text as the unperturbed orbit), which is given by

$$u_i'' + \Omega^2 u_i = 0; \tag{8}$$

and in integrating numerically only the small change of coordinates u_i with respect to the unperturbed orbit caused by the perturbation force.

Laying down formula

$$u_i = (\alpha_{i0} + \Delta\alpha_i)\sin\Omega s + (\beta_{i0} + \Delta\beta_i)\cos\Omega s\,, \qquad t = \int (r_0 + \Delta r)\,ds = t_0 + \Delta t\,, \tag{9}$$

where the Δ values designate the perturbation deviations, there is obtained from (5) a system of 10 first-order differential equations

$$\left.\begin{aligned}
(\Delta\alpha_i)' &= -\frac{1}{\Omega} F_i \sin\Omega s \quad (i = 1, 2, 3, 4)\\
(\Delta\beta_i)' &= +\frac{1}{\Omega} F_i \cos\Omega s \quad (i = 1, 2, 3, 4)\\
(\Delta t)' &= \Delta r\\
A' &= \sum_{j=1}^{4} S_j u_j'\,,
\end{aligned}\right| \tag{10}$$

where $F_i = r/4\, S_i + A/2\, u_i$. Analoguously, (6) yields 9 first-order differential equations

$$\left.\begin{aligned}
(\Delta\alpha_i)' &= -\frac{1}{\Omega} G_i \sin\Omega s \quad (i = 1, 2, 3, 4)\\
(\Delta\beta_i)' &= +\frac{1}{\Omega} G_i \cos\Omega s \quad (i = 1, 2, 3, 4)\\
(\Delta t)' &= r\left[\frac{2}{M}\left(r\,\Omega^2 + \sum_{j=1}^{4} (u_j')^2\right)\right]^{1/2} - r_0\,.
\end{aligned}\right| \tag{11}$$

where, accordingly, the values of G_i are equal to the right-hand side of (6). In order to test the above-described method in practice in orbit calculations, it was programmed for an electronic computer; the

the integration of differential equations (10) and (11) is carried out by the 4th-order Runge-Kutta method. Perturbation by a third body and perturbation by the flattening of the central body were dealt with as examples.

II. Perturbation by a Third Body

In addition to the central body with a mass of M and the vehicle, a perturbation body with a mass of m_1 shall be present. The perturbation acceleration upon the vehicle then assumes the form of

$$\boldsymbol{S} = -m_1 \left(\frac{\boldsymbol{r} - \boldsymbol{r}_1}{|\boldsymbol{r} - \boldsymbol{r}_1|^3} + \frac{\boldsymbol{r}_1}{r_1^3} \right) \tag{12}$$

if r_1 designates the location of the perturbation body.

Since the vehicle is massless, the perturbation body describes a Kepler orbit around M with the equation of motion

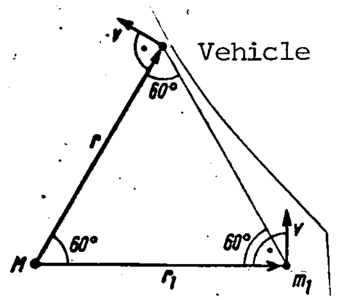
$$\ddot{\boldsymbol{r}}_1 = \frac{m_1 + M}{r_1} \boldsymbol{r}_1 . \tag{13}$$

In the program, the coordinate system x_1, x_2, x_3 was so chosen that the perturbation body remains in the x_1, x_2 plane and the large axis of its orbit ellipse lies on the x_1 axis. Then entering its mass, initial position and eccentricity suffices for determination of its orbit. The vehicle is entered in terms of its initial position and initial velocity at time t = 0. According to choice, the program intergrates system (10) or (11), where the step width in s during integration is kept constant. Due to the regularization, the ejection of the vehicle / from the central body can also be treated; in this case, instead of the initial velocity the location of the point of reversal of the unperturbed vehicle orbit is entered.

1. Example

Central body: earth, perturbation body: moon on circular orbit.

As test, the motion was calculated of the vehicle which is located in the Lagrage point L_4, and the initial velocity of which is perpendicular to r and is equal to the moon velocity.



M = earth mass

m_1 = moon mass

r_1 = 384400 km,

v = 88587,38 km/ day

While the unperturbed orbit is an ellipse with M as the focal point, the perturbed orbit must be a circular orbit around M (i.e. r= const)., where angle φ(= angle between r and r_1) is constant = 60°. In the following table, according to equation system (10) a step width of 10-6 day/km in s was used in the computation. This means approximately 70 steps per revolution of the mobile:

One revolution lasts approximately 27 days. - Table 1 -

Zeit (Tage) 1.	gestörte Bahn r^2 2.	tgφ	ungestörte Bahn r^2 3.	tgφ
0	14776336	1,7320508	14776336	1,7320508
1,5376000	14776336	1,7320508	14798660	1,7316929
3,0752000	14776336	1,7320509	14863020	1,7292456
4,6128000	14776337 · 10⁴	1,7320509	14961850 · 10⁴	1,7229132
6,1504001	14776338	1,7320508	15083434	1,7114590
7,6880003	14776339	1,7320507	15213023	1,6943905
9,2256004	14776340	1,7320504	15335418	1,6720128

1= time (days). 2= perturbed orbit. 3= unperturbed orbit.

In this case a comparison with the calculation according to equation system (11) is particularly simple; since the vehicle moves in a circular orbit, the root in (7) is constantly equal to 1; thus the s in (10) and (11) is the same. Thus, for the same step width there results in s the same step width in time. Indeed, a computation according to (11) yielded exactly the same numbers as in Table 1. One integration step requires approximately the same machine time in both cases. Calculations with different step widths yielded the result that the variation still present in the perturbed r is not due to the inaccuracy of the integration, but is due to the error in the initial position and velocity of the vehicle.

2nd Example

Ejection of the vehicle from the central body earth in the x_3 direction.

Perturbation body: moon in circular orbit, M = earth mass, m_1 = = moon mass, r_1 = 384 400 km. apogee of the vehicle at X_2 = 200 000 km. Step width 10-6 day/km. After 3.64 days the mobile returns approximately to the starting point.

Table 2

Maximum deviation from the unperturbed orbit	Maximum approach to the Center of the earth
in first revolution 180 km.	after the first revolution 1/8km.
in second revolution 230 km.	after the second revolution 1/3km.
(Each time at the return from the apogee to the earth)	

In spite of the very close approach to the earth, no considerable integration errors occur due to the regularization in the vicinity of earth, as is proved by calculations with different step widths.

(Details with respect to step widths in IV).

III. Perturbation By the Flattening of the Central Body

The gravitation potential of the central body, the equatorial plane of which is assumed to be in the x_1, x_2 plane, shall be a function only of the geographical latitude, i.e. the potential has the following form

$$V = \frac{M}{r}\left(1 - \sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k\left(\frac{x_3}{r}\right)\right)$$

where

$$\begin{aligned} R &= \text{equator radius of the central body.} \\ P_k &= \text{Legendre polynomials} \\ J_k &= \text{flattening coefficients.} \end{aligned} \tag{14}$$

The values of J_k can be determined from the orbits of the satellites circling the earth; one of the latest calculations (up to J_{12}) can be found in /4/.

If the perturbation potential is designated by V_s ,

$$V_s = -\frac{M}{r}\sum_{k=2}^{\infty} J_k \left(\frac{R}{r}\right)^k P_k\left(\frac{x_3}{r}\right). \tag{15}$$

the perburbation acceleration is obtained as a gradient of V_s and the transformed equation of motion corresponding to (5) reads

$$u_i'' + \Omega^2 u_i = \frac{r}{4}\frac{\partial V_s}{\partial u_i} + \frac{V_s - V_{s0}}{2} u_i \tag{16}$$

with

V_{so} = perturbation potential at time $t = 0$.

Since the perturbation force can here be derived from a potential and the work can thus be expressed as a difference of potential, a system of only 9 differential equations is obtained in formulation (9); (6) does not have to be considered, since it no longer contributes to economy in integration.

The vehicle is again entered into the program in terms of its initial velocity, the central body is entered with its mass, the equator radius and the coefficients J_k.

3rd Example

Calculation of the rotation of the nodal-point and of the perigee point in the orbit plane and comparison with the corresponding analytical formulas.

For reasons of simplicity, only the flattening coefficient J_2 was assumed different from 0:

$$J_2 = 0.00108292 \text{ (= Value for earth, see /4/)}, \; J_k = 0 \text{ for } k > 2.$$

In the same manner, the values of the earth were inserted for M and R.

a) initial values of the vehicle: radius vector r_A = 9000 km, velocity v_A = 579827.58 km/day, perpendicularly to r_A. From this results an eccentricity of $e \approx 0.0146$.

Disregarding e^2 and higher powers of e (compare/5/, p. 75), the formula, adapted for this case, for the secular rotation of the nodal is

$$\frac{d\Omega}{dt} = -\cos\iota\, r_A^{-4} v_A^{-1} (1 - 3e) \frac{3}{2} J_2 M R^2, \quad \Delta\Omega \approx \frac{d\Omega}{dt} T, \tag{17}$$

L = angle between v_A and the equator is plane, T= time of revolution of the vehicle.

A comparison for 3 different inclinations L yielded

Table 3

$\cos\iota$	$\Delta\Omega$ ac. (17)	$\Delta\Omega$ with Program (50 steps/revolution)
0,70710678	− 0,003521	− 0,00352
0,08623253	− 0,000429	− 0,00043
0,99962562	− 0,004978	− 0,00494

c) Initial values of the vehicle: r_A = 36 000 km, v_A = 289913.79 km/day, perpendicularly to r_A. This results in the same eccentricity as in a).

A comparison for $\iota = \pi/4$ yielded

Table 4

$\Delta\Omega$ ac. (17)	$\Delta\Omega$ with Program (50 steps/revolution)
− 0,0002204	− 0,000220

c) initial values of the vehicle as in a). According to /5/

$$\frac{d\omega}{dt} = (5\cos^2\iota - 1)\, r_A^{-4}\, v_A^{-1}\, (1 - 3e)\, \frac{3}{4}\, J_2 M R^2 . \tag{18}$$

is obtained for the rotation of the perigee in this case.

Since it is difficult to read this rotation from Cartesian coordinates for the relatively large step width, only the Two cases were calculated. A comparison with (18) yielded cosign L.

Table 5.

	1.	2.
$\cos\iota$	$\Delta\omega$ nach (18)	$\Delta\omega$ mit Programm
1	0,00995	0,0098
0	− 0,00248	− 0,0026

1= according to. 2= with program.

The deviations are in accordance with the reading accuracy of the perigee from the Cartesian coordinates.

VI. Comparison with the Encke Method

The Encke Method is also a perturbation method; however, it works without regularization. Starting from (4) and (12) the defferential equations

$$\ddot{\delta r} = M\left(\frac{r_0}{r_0^3} - \frac{r}{r^3}\right) - m_1\left(\frac{r - r_1}{|r - r_1|^3} + \frac{r_1}{r_1^3}\right) \tag{19}$$

are obtained for coordinate perturbations $\delta r - r_o$, if r designates the perturbed and r the unperturbed orbit. As above, the subscript 1 refers to pertubation body. Equations (19), written as a system of 6 1st - order differential equations, yields

$$\delta\dot{v} = M\left(\frac{r_0}{r_0^3} - \frac{r_0 + \delta r}{|r_0 + \delta r|^3}\right) - m_1\left(\frac{r_0 + \delta r - r_1}{|r_0 + \delta r - r_1|^3} + \frac{r_1}{r_1^3}\right), \quad \delta\dot{r} = \delta v . \tag{20}$$

In the program for the Encke method, this system is again integrated according to RUNGE-KUTTA (4th order). The step width in time t is

either: (21) kept constant,

or: (22) varied in such a way that the corresponding step width in $\dot{s}$ is constant,

or: (23) automatically regulated by the error in integration.

Determination of the unperturbed vehicle orbit r_o is conducted according to the method of STUMPFF /6/; the equation coorexponding to the Kepler equation, occurring there, is solved iteratively.

Comparison of the machine times for one integration step:

Table 6

Encke	0.2 Sec
KS method (with (10)or (11)	0.6 Sec

4th Example

Initial location and initial velocity of the vehicle: $x = 0$, $y = 0$, $z = 10000$ km, $v_x = 0$, $v_y = 750000$ km/day, $v_z = 0$, central mass = moon in curcular orbit around the earth at a distance of 384400 km.

These initial conditions result in a very eccentric orbit ($e = 0.94$); it is thus to be expected that the example speaks in favor of the regularizing KS method. The time at which comparison is made is t= 3.1841455 days, i.e. after a little more than one revolution of the vehicle.

Table 7

Method	No. of steps	step width ds or dt	x	y	z
KS with (10) or (11)	2	$ds = 2\cdot10^{-5}$	60,00	35379,12	−33888,55
	8	$5\cdot10^{-6}$	80,98	35400,52	−33911,38
	40	$1\cdot10^{-6}$	80,99	35400,52	−33911,34
	200	$2\cdot10^{-7}$	80,99	35400,52	−33911,34
Encke mass (21)	16	$dt = 0{,}2$	26,47	62365,72	−37286,08
	64	0,05	15,58	22144,64	−36560,63
	319	0,01	81,03	35439,95	−33960,81
	1593	0,002	80,99	35400,74	−33911,40
Encke mass (22)	8	$ds = 5\cdot10^{-6}$	0,35	49464 46	−4029,43
	40	$1\cdot10^{-6}$	81,06	35432,85	−33994,81
	200	$2\cdot10^{-7}$	80,99	35400,85	−33911,42
	800	$5\cdot10^{-8}$	80,99	35400,52	−33911,35
Encke mass (23)	46	$0{,}0047 < dt < 0{,}152$	81,00	35404,30	−33915,00
	103	$0{,}00119 < dt < 0{,}152$	80,99	35400,64	−33911,33
	272	$0{,}00059 < dt < 0{,}038$	80,99	35400,50	−33911,35
	356	$0{,}00040 < dt < 0{,}025$	80,99	35400,52	−33911,35

As can be seen here, the Encke method with constant step width in t (21) is exceedingly poor: with 1593 steps it is still less accurate than the KS method with g steps. Within the scope of the Encke methods, that with the automatic step-width adaption (23) is the best; this one must therefore be compared with the KS method:

Table 8

1. Accuracy in x,y,z, 2. KS method number of steps 3. calculation time

4. Encke method with (23) number of steps 5. calculation time

1. Genauigkeit in x, y, z	2. KS-Methode Anzahl Schritte	3. Rechenzeit	4. Encke-Methode mit (23) Anzahl Schritte	5. Rechenzeit
± 0,1	8	4,8 sec	103	20,6 sec
± 0,01	40	24 sec	356	71,2 sec

Concerning the last column of Table 8 it must be remarked that the calculation times are in reality still approximately 20 to 30 percent greater, since for the step number only those steps are counted which were used for integration; in the case of the automatic step-width adaptation, Runge-Kutta steps are in addition required, which only serve for testing of the accuracy.

5th Example

Initial location and initial velocity of the vehicle: x = 0, y = o, z = 75000 km, v = 0, v_y = 200000 km/day, v_z = 0. Remaining data as in example 4.

Under these conditions the vehicle almost describes a circular orbit (e = 0.007); thus the Encke method is preferable here. Time of comparison is t = 3.0176050 days, i.e. after approximately 5/4 revolutions of the vehicle.

1. Method

2. number steps

3. step width ds or dt

4. KS with (10) or (11)

Table 9.

5. Encke with (21)

1. Methode	2. Anzahl Schritte	3. Schrittweite ds bzw. dt	x	y	z
4. KS mit (10) oder (11)	2	$ds = 2 \cdot 10^{-6}$	−6,31	75162,85	−7502,45
	8	$5 \cdot 10^{-6}$	4,37	75171,71	−7510,35
	40	$1 \cdot 10^{-6}$	4,34	75171,72	−7510,34
	200	$2 \cdot 10^{-7}$	4,34	75171,72	−7510,34
5. Encke mit (21)	16	$dt = 0{,}2$	4,33	75173,26	−7508,02
	61	0,05	4,34	75171,72	−7510,33
	302	0,01	4,34	75171,72	−7510,34
	1509	0,002	4,34	75171,72	−7510,34

Since the mobile orbit is almost a circle and the perturbation force has always approximately the same order of magnitude, in this example the Encke methods with (22) and (23) are not better than those with constant step width in t (21).

As can be seen from Table 9, fewer steps are required for the same accuracy with the KS method than with Encke in the case of the circular orbit: 16 steps of the second method are considerably less accurate than g steps of the first. The ratio of the step numbers for the same accuracy can be given at approximately 1 : 4. If the machine times for one step (see Table 6) are also considered, the Encke method thus, even in the case most favorable for it, tends to be inferior to the KS method.